ON THE INVARIANT UNIFORM ROE ALGEBRA

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ABSTRACT. Let Γ be a countable discrete group. We show that Γ has the approximation property if and only if Γ is exact and for any operator space $S \subseteq \mathcal{K}(H)$ we have
$$C_u^*(\Gamma)^\Gamma \otimes S = (C_u^*(\Gamma) \otimes S)^\Gamma,$$
where $C_u^*(\Gamma)$ is the uniform Roe algebra with the right adjoint Γ-action. This answers a question of J. Zacharias. We also show that characterisations of several properties of Γ in terms of the reduced group C^*-algebra $C_\lambda^*(\Gamma)$ apply to the invariant uniform Roe algebra $C_u^*(\Gamma)^\Gamma$.

1. INTRODUCTION

Let Γ be a countable discrete group and let $l^2\Gamma$ denote the Hilbert space of square-summable functions on Γ with orthonormal basis $\{\delta_s \mid s \in \Gamma\}$. We identify operators $T \in \mathcal{B}(l^2\Gamma)$ with the $(\Gamma \times \Gamma)$-matrix given by $[\langle T\delta_s, \delta_t \rangle]_{s,t \in \Gamma}$.

Let λ and ρ denote the left and right regular representations of Γ on $l^2\Gamma$ given by

$$\lambda(g)\delta_s := \delta_{gs} \quad \text{and} \quad \rho(g)\delta_s := \delta_{sg^{-1}}, \quad g, s \in \Gamma,$$

respectively. The *group C^*-algebra* $C_\lambda^*(\Gamma)$ of Γ is the C^*-algebra generated by $\lambda(\Gamma) \subseteq \mathcal{B}(l^2\Gamma)$.

In this paper, we consider two C^*-subalgebras of $\mathcal{B}(l^2\Gamma)$ containing $C_\lambda^*(\Gamma)$. The first is the *uniform Roe algebra* $C_u^*(\Gamma)$. This algebra may be thought of as the closure of scalar $(\Gamma \times \Gamma)$-matrices $[\alpha_{s,t}]$ of finite width (i.e. $\{st^{-1} \in \Gamma \mid \alpha_{s,t} \neq 0\}$ is finite) with uniformly bounded entries acting on $l^2\Gamma$. The second is the *group von Neumann algebra* $L(\Gamma) := C_\lambda^*(\Gamma)''$. By a fundamental result of Murray and von Neumann (c.f. [BO08, Secion 6.1]), we have $L(\Gamma) = \rho(\Gamma)'$, the commutant of $\rho(\Gamma)$. It follows that $L(\Gamma)$ can be identified with the algebra of elements in $\mathcal{B}(l^2\Gamma)$ that are constant down the diagonals i.e. satisfying $\alpha_{s,t} = \alpha_{st^{-1},e}$ for s, $t \in \Gamma$.

The adjoint action $\mathrm{Ad}(\rho)$ of Γ on $\mathcal{B}(l^2\Gamma)$ preserves $C_u^*(\Gamma)$ and the invariant elements are precisely the intersection:

$$C_u^*(\Gamma)^\Gamma = C_u^*(\Gamma) \cap L(\Gamma).$$

Clearly, we have

$$C_\lambda^*(\Gamma) \subseteq C_u^*(\Gamma)^\Gamma.$$

Definition 1.1 (cf. [Roe03, 11.5.3]). We say that Γ has the *invariant translation approximation property* (ITAP) if

$$C_\lambda^*(\Gamma) = C_u^*(\Gamma)^\Gamma.$$

Conjecturally, all countable discrete groups have the ITAP. It is proved in [Roe03] that *amenable* groups and *finitely generated free* groups have the ITAP.

Haagerup and Kraus defined and studied the *approximation property* (AP) for a group, which we recall below briefly for the convenience of the reader. See [HK94, BO08] for more details.

Let $A(\Gamma) = A_2(\Gamma)$ denote the Fourier algebra of Γ and let $B_2(\Gamma) = M_0 A(\Gamma)$ denote the algebra of completely bounded multipliers of $A(\Gamma)$. Let $Q(\Gamma)$ denote the completion of $l^1\Gamma$ in the norm

$$\|f\|_Q := \sup\left\{\left|\sum f(s)u(s)\right| \mid u \in B_2(\Gamma), \|u\|_{B_2} \leq 1\right\}.$$

Then $B_2(\Gamma) \cong Q(\Gamma)^*$.

Definition 1.2 (Haagerup-Kraus, [HK94, Definition 1.1])**.** We say that Γ has the *approximation property* (AP) if the constant function $1 \in B_2(\Gamma)$ is in the $\sigma(B_2(\Gamma), Q(\Gamma))$-closure of $A(\Gamma)$.

Example 1.3.   (i) Weakly amenable, in particular amenable, groups have the AP (cf. [HK94]).
   (ii) $SL_3(\mathbb{Z})$ does not have the AP (cf. [LDlS11]). More generally, lattices in connected simple Lie groups with finite center and real rank ≥ 2 do not have the AP (cf. [HdL12]).

In [Zac06], Zacharias proved that groups with the AP have the ITAP. In fact, he proved the following stronger statement. We fix an infinite dimensional separable Hilbert space H. The C^*-algebra of all bounded (respectively, compact) linear operators on H is denoted by $\mathcal{B}(H)$ (respectively, $\mathcal{K}(H)$).

Let $S \subseteq \mathcal{B}(H)$ be a (concrete) operator space. Clearly, we have inclusions

$$C^*_\lambda(\Gamma) \otimes S \subseteq C^*_u(\Gamma)^\Gamma \otimes S \subseteq (C^*_u(\Gamma) \otimes S)^\Gamma,$$

where Γ is acting trivially on S.

Recall that a countable discrete group Γ said to be *exact* if $C^*_\lambda(\Gamma)$ is exact (cf. [KW99, BO08]).

Theorem 1.4 (Zacharias [Zac06, Theorem 3.2])**.** *Let Γ be a countable discrete exact group. Then the following are equivalent.*

   (i) The group Γ has the AP.
   (ii) The equality $C^*_\lambda(\Gamma) \otimes S = (C^*_u(\Gamma) \otimes S)^\Gamma$ holds for all operator spaces $S \subseteq \mathcal{K}(H)$.
   (iii) The equality $C^*_\lambda(\Gamma) \otimes S = (C^*_u(\Gamma) \otimes S)^\Gamma$ holds for all operator spaces $S \subseteq \mathcal{B}(H)$.

Remark 1.5.   (i) Groups with the AP are exact by [HK94, Theorem 2.1] and the discussion following it.
   (ii) Group Γ has the ITAP if and only if the identity $C^*_\lambda(\Gamma) \otimes S = C^*_u(\Gamma)^\Gamma \otimes S$ holds for all operator spaces $S \subseteq \mathcal{K}(H)$ if and only if the identity $C^*_\lambda(\Gamma) \otimes S = C^*_u(\Gamma)^\Gamma \otimes S$ holds for all operator spaces $S \subseteq \mathcal{B}(H)$. This is trivially true.

At the end of [Zac06], Zacharias asked the following:

Question 1.6. Is it true that if Γ is a discrete group, then

$$C^*_u(\Gamma)^\Gamma \otimes S = (C^*_u(\Gamma) \otimes S)^\Gamma$$

for all operator spaces $S \subseteq \mathcal{K}(H)$?

Our main result is the following, which answers Question 1.6 in the negative.

Theorem 1.7. *Let Γ be a countable discrete exact group. Then the following are equivalent.*

   (1) The group Γ has the AP.
   (2) The equality $C^*_u(\Gamma)^\Gamma \otimes S = (C^*_u(\Gamma) \otimes S)^\Gamma$ holds for all operator spaces $S \subseteq \mathcal{K}(H)$.
   (3) The equality $C^*_u(\Gamma)^\Gamma \otimes S = (C^*_u(\Gamma) \otimes S)^\Gamma$ holds for all operator spaces $S \subseteq \mathcal{B}(H)$.

Since $SL_3(\mathbb{Z})$ does not have the AP by [LDlS11], we get the following as a corollary.

Corollary 1.8. *Let $\Gamma = SL_3(\mathbb{Z})$. Then there exists an operator space $S \subseteq \mathcal{K}(H)$ such that*

$$C^*_u(\Gamma)^\Gamma \otimes S \neq (C^*_u(\Gamma) \otimes S)^\Gamma.$$

$\square$

We also show that various approximation properties hold for $C_\lambda^*(\Gamma)$ if and only if they hold for $C_u^*(\Gamma)^\Gamma$. See Theorem 3.1 for the precise statement.

2. Proof of the Main Theorem (1.7)

2.1. **The Slice Map Property.** We start with some definitions. All tensor products we consider are spatial (minimal) ones.

Definition 2.1 (cf. [Was76]). Let $T \subseteq A$ and $S \subseteq B$ be operator spaces. The *Fubini product* $F(T, S, A \otimes B)$ of T and S is defined as the set of all $x \in A \otimes B$ such that $(\phi \otimes \mathrm{id}_B)(x) \in S$ for all $\phi \in A^*$ and $(\mathrm{id}_A \otimes \psi)(x) \in T$ for all $\psi \in B^*$.

Remark 2.2. We always have

$$T \otimes S \subseteq F(T, S, A \otimes B) = F(T, B, A \otimes B) \cap F(A, S, A \otimes B) \subseteq A \otimes B. \qquad (1)$$

Definition 2.3. We say that $(T, S, A \otimes B)$ has the *slice map property* if

$$F(T, S, A \otimes B) = T \otimes S.$$

We say that A has the slice map property for B if $(A, S, A \otimes B)$ has the slice map property for all operator spaces $S \subseteq B$.

The following is a simple but key result that connects fixed points to the slice map property.

Proposition 2.4. *Let Γ be a countable discrete group and let S and T be operator spaces. Suppose that Γ is acting on T by completely bounded maps. Letting Γ act trivially on S, we get a Γ-action on $T \otimes S$. Then we have*

$$F(T^\Gamma, S, T \otimes S) = (T \otimes S)^\Gamma.$$

Proof. Let $x \in T \otimes S$. Then for any $\psi \in S^*$ and $g \in \Gamma$, we have

$$g((\mathrm{id}_T \otimes \psi)(x)) = (\mathrm{id}_T \otimes \psi)(g \otimes \mathrm{id}_S)(x). \qquad (2)$$

Hence if $x \in (T \otimes S)^\Gamma$, then $(\mathrm{id}_T \otimes \psi)(x) \in T^\Gamma$ for all $\psi \in S^*$. It follows that $(T \otimes S)^\Gamma \subseteq F(T^\Gamma, S, T \otimes S)$. Conversely, if $x \in F(T^\Gamma, S, T \otimes S)$, then for any $\psi \in S^*$ and $g \in \Gamma$, we have $(\mathrm{id}_T \otimes \psi)(x - (g \otimes \mathrm{id}_S)(x)) = 0$ by (2). Since $\mathrm{id}_T \otimes S^*$ separates points of $T \otimes S$, we see that $x \in (T \otimes S)^\Gamma$. $\qquad \square$

Corollary 2.5. *Let $S \subseteq \mathcal{B}(H)$ be an operator space. Then $C_u^*(\Gamma)^\Gamma \otimes S = (C_u^*(\Gamma) \otimes S)^\Gamma$ if and only if $(C_u^*(\Gamma)^\Gamma, S, C_u^*(\Gamma) \otimes S)$ has the slice map property.* $\qquad \square$

Lemma 2.6. *Let $T \subseteq A$ and $S \subseteq B$ be operator spaces. Suppose that $(A, S, A \otimes B)$ has the slice map property. Then*

$$F(T, S, A \otimes S) = F(T, S, A \otimes B)$$

Proof. By assumption, we have

$$A \otimes S = F(A, S, A \otimes B).$$

On the other hand, any $\psi \in S^*$ extends to an element of B^*, hence

$$F(T, S, A \otimes S) = (A \otimes S) \cap F(T, B, A \otimes B).$$

It follows that $F(T, S, A \otimes S) = F(T, S, A \otimes B)$, by (1). $\qquad \square$

Corollary 2.7. *Let $T \subseteq A$ and $S \subseteq B$ be operator spaces. Suppose that $(A, S, A \otimes B)$ and $(T, S, A \otimes S)$ have the slice map property. Then $(T, S, T \otimes B)$ also has the slice map property.*

Proof. Follows from Lemma 2.6 and the following commutative diagram of inclusions:

$$
\begin{array}{ccc}
F(T,S,A \otimes S) & \hookrightarrow & F(T,S,A \otimes B) \\
\cup \uparrow & & \uparrow \cup \\
T \otimes S & \hookrightarrow & F(T,S,T \otimes B)
\end{array}
$$

$\square$

Theorem 2.8. *Let Γ be a countable* exact *discrete group. If the equality*

$$C_u^*(\Gamma)^\Gamma \otimes S = (C_u^*(\Gamma) \otimes S)^\Gamma$$

holds for all operator spaces $S \subseteq \mathcal{K}(H)$, then $C_u^*(\Gamma)^\Gamma$ has the slice map property for $\mathcal{K}(H)$.

Proof. Let $T := C_u^*(\Gamma)^\Gamma$, $A := C_u^*(\Gamma)$ and $B := \mathcal{K}(H)$. By [Oza00, Theorem 3], the algebra A is nuclear and therefore has the OAP (see Definition 2.9). From [Kra91, Theorem 5.5] we see that $(A, S, A \otimes B)$ has the slice map property for all operator spaces $S \subseteq B$. By Corollary 2.5, the triple $(T, S, A \otimes S)$ has the slice map property for all operator spaces $S \subseteq B$. Now Corollary 2.7 completes the proof. $\square$

2.2. **Operator Approximation Property.** Now we relate the slice map property to the approximation property.

Definition 2.9 (cf. [ER90]). A completely bounded map $\theta \colon A \to B$ of operator spaces is said to have the *operator approximation property* (OAP) if there is a net of continuous finite-rank maps $F_n \colon A \to B$ converging to θ in the stable point-norm topology: $F_n \otimes \mathrm{id}_{\mathcal{K}(H)}(a) \to \theta \otimes \mathrm{id}_{\mathcal{K}(H)}(a)$ for all $a \in A \otimes \mathcal{K}(H)$ as $n \to \infty$. We say that an operator space A has the OAP if $\mathrm{id}_A \colon A \to A$ has the OAP.

By [Kra91, Theorem 5.5], a C^*-algebra has the OAP if and only if it has the slice map property for $\mathcal{K}(H)$. Combining with Theorem 2.8, we get the following.

Corollary 2.10. *Let Γ be a countable* exact *discrete group. If the equality*

$$C_u^*(\Gamma)^\Gamma \otimes S = (C_u^*(\Gamma) \otimes S)^\Gamma$$

holds for all operator spaces $S \subseteq \mathcal{K}(H)$, then $C_u^*(\Gamma)^\Gamma$ has the OAP. $\square$

Finally, by a result of Haagerup and Kraus, a countable discrete group Γ has the AP if and only if $C_\lambda^*(\Gamma)$ has the OAP (cf. [HK94, Theorem 2.1]). A simple inspection shows that the proof of [HK94, Theorem 2.1(c) $\Rightarrow$ (a)] actually demonstrates the following.

Proposition 2.11. *Let Γ be a countable discrete group and let $\imath_\Gamma \colon C_\lambda^*(\Gamma) \to L(\Gamma)$ denote the inclusion map. If $\imath_\Gamma$ has the OAP, then Γ has the AP.* $\square$

Now the following corollary is immediate.

Corollary 2.12. *The algebra $C_\lambda^*(\Gamma)$ has the OAP if and only if $C_u^*(\Gamma)^\Gamma$ has the OAP if and only if Γ has the AP.*

Proof. If $C_\lambda^*(\Gamma)$ or $C_u^*(\Gamma)^\Gamma$ has the OAP, then so does $\imath_\Gamma$ and thus Γ has the AP by Proposition 2.11. Conversely, if Γ has the AP, then $C_\lambda^*(\Gamma)$ has the OAP by [HK94, Theorem 2.1] and $C_u^*(\Gamma)^\Gamma = C_\lambda^*(\Gamma)$ by Theorem 1.4. $\square$

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. In view of Theorem 1.4, it suffices to prove (2) $\Rightarrow$ (1). However, this is clear from Corollary 2.10 and Corollary 2.12. $\square$

3. Other Approximation Properties

We end with a theorem which is in agreement with the conjecture that all countable discrete groups have the ITAP. See [BO08] for the terminologies used. Compare Corollary 2.12.

Theorem 3.1. *Let Γ be a countable discrete group. The following statements hold.*

(i) *The algebra $C_\lambda^*(\Gamma)$ is nuclear if and only if $C_u^*(\Gamma)^\Gamma$ is nuclear if and only if Γ is amenable.*
(ii) *The algebra $C_\lambda^*(\Gamma)$ has the CBAP if and only if $C_u^*(\Gamma)^\Gamma$ has the CBAP if and only if Γ is weakly amenable.*
(iii) *The algebra $C_\lambda^*(\Gamma)$ has the SOAP if and only if $C_u^*(\Gamma)^\Gamma$ has the SOAP if and only if Γ has the AP.*
(iv) *The algebra $C_\lambda^*(\Gamma)$ is exact if and only if $C_u^*(\Gamma)^\Gamma$ is exact if and only if Γ is exact.*

The proof is based on the following analogue of Proposition 2.11.

Lemma 3.2. *Let Γ be a countable discrete group.*

(i) *The inclusion $\imath_\Gamma \colon C_\lambda^*(\Gamma) \to L(\Gamma)$ is nuclear if and only if Γ is amenable.*
(ii) *The inclusion $\imath_\Gamma \colon C_\lambda^*(\Gamma) \to L(\Gamma)$ has the CBAP if and only if Γ is weakly amenable.*
(iii) *The inclusion $\imath_\Gamma \colon C_\lambda^*(\Gamma) \to L(\Gamma)$ has the SOAP if and only if Γ has the AP.*

Proof.   (i) Follows from the proof of [BO08, Theorem 2.6.8(9) $\Leftrightarrow$ (1)].
  (ii) Follows from the proof of [BO08, Theorem 12.3.10].
  (iii) Follows from Proposition 2.11 and [HK94, Theorem 2.1].

$\square$

Proof of Theorem 3.1. As remarked in Example 1.3, amenable and weakly amenable groups have the AP, thus have the ITAP by Theorem 1.4. Combining with Lemma 3.2, we get the statements (i), (ii) and (iii).

Finally, the statement (iv) follows from [Oza00, Theorem 3], since exactness passes to subalgebras. $\square$

References

[BO08] Nathanial P. Brown and Narutaka Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008. MR 2391387 (2009h:46101)

[ER90] Edward G. Effros and Zhong-Jin Ruan, *On approximation properties for operator spaces*, Internat. J. Math. **1** (1990), no. 2, 163–187. MR 1060634 (92g:46089)

[HdL12] Uffe Haagerup and Tim de Laat, *Simple lie groups without the approximation property*, http://arxiv.org/abs/1201.1250, January 2012.

[HK94] Uffe Haagerup and Jon Kraus, *Approximation properties for group C^*-algebras and group von Neumann algebras*, Trans. Amer. Math. Soc. **344** (1994), no. 2, 667–699. MR 1220905 (94k:22008)

[Kra91] Jon Kraus, *The slice map problem and approximation properties*, J. Funct. Anal. **102** (1991), no. 1, 116–155. MR 1138840 (92m:47083)

[KW99] Eberhard Kirchberg and Simon Wassermann, *Exact groups and continuous bundles of C^*-algebras*, Math. Ann. **315** (1999), no. 2, 169–203. MR 1721796 (2000i:46050)

[LDlS11] Vincent Lafforgue and Mikael De la Salle, *Noncommutative L^p-spaces without the completely bounded approximation property*, Duke Math. J. **160** (2011), no. 1, 71–116. MR 2838352

[Oza00] Narutaka Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 8, 691–695. MR 1763912 (2001g:22007)

[Roe03] John Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003. MR 2007488 (2004g:53050)

[Was76] Simon Wassermann, *The slice map problem for C^*-algebras*, Proc. London Math. Soc. (3) **32** (1976), no. 3, 537–559. MR 0410402 (53 #14152)

[Zac06] Joachim Zacharias, *On the invariant translation approximation property for discrete groups*, Proc. Amer. Math. Soc. **134** (2006), no. 7, 1909–1916 (electronic). MR 2215118 (2007f:46056)

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